

# McClure-Smith cosimplicial machinery and the cacti operad

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## Abstract

McClure and Smith constructed a functor that sends a topological multiplicative operad  $\mathcal{O}$  to an  $E_2$  algebra  $\text{Tot}\mathcal{O}^\bullet$ . They define in fact an operad  $\mathcal{D}_2$  (acting on the totalization  $\text{Tot}\mathcal{O}^\bullet$ ) weakly equivalent to the little 2-disks operad. On the other hand, Salvatore showed that  $\mathcal{D}_2$  is isomorphic to the cacti operad  $MS$ , which has a nice geometric description. He also built a geometric action of  $MS$  on  $\text{Tot}\mathcal{O}^\bullet$ . In this paper we detail the McClure-Smith action and the cacti action. Our main result says that they are compatible in the sense that some squares must commute.

## 1 Introduction

A *multiplicative operad* is a topological nonsymmetric operad  $\mathcal{O}$  endowed with a morphism  $\mathcal{A}s \rightarrow \mathcal{O}$  from the associative operad  $\mathcal{A}s = \{*\}_{n \geq 0}$  to  $\mathcal{O}$ . To any multiplicative operad, McClure and Smith [1, Section 10] associate a cosimplicial space  $\mathcal{O}^\bullet$ . They also define an  $E_2$  operad  $\mathcal{D}_2$  (see Definition 4.1), which acts on the totalization  $\text{Tot}\mathcal{O}^\bullet$ . This gives a functor from the category of multiplicative operads to the category of  $E_2$ -algebras (for us, an  $E_2$ -algebra is a topological space endowed with an action of an operad weakly equivalent to the little 2-disks operad).

On the other hand, Salvatore [2, Proposition 8.2] shows that the cacti operad  $MS$  (see Definition 3.9), which has a nice geometric description, is isomorphic to  $\mathcal{D}_2$ . He also proves [2, Theorem 5.4] that  $MS$  acts on the totalization  $\text{Tot}\mathcal{O}^\bullet$ . The idea of his proof is as follows. To any cactus, elements  $a_1^\bullet, \dots, a_n^\bullet \in \text{Tot}\mathcal{O}^\bullet$ , and  $t \in \Delta^k$ , he associates a planar tree whose vertices (except the root and the leaves) are labelled by the entries of  $\mathcal{O}$ . Using now the operad structure of  $\mathcal{O}$ , he gets an operation  $\theta \in \mathcal{O}^k$ .

We will explicitly construct  $\theta$ , without using trees, and our proof will be more combinatorial. In fact, from the same data as those of Salvatore, we first associate a word (instead of a labelled tree) on the alphabet  $\bar{n} = \{1, \dots, n\}$ . Next, by "suitable cutting" this word, we obtain an explicit formula for  $\theta$  (many illustrative examples are given).

The natural question one can ask is whether the McClure-Smith action and the cacti action are equivalent. The following theorem, which is the main result of this paper, gives a positive answer.

**Theorem 1.1.** *Let  $\mathcal{O}^\bullet$  be cosimplicial space associate to a multiplicative operad. Then the McClure-Smith and the cacti actions on the totalization  $\text{Tot}\mathcal{O}^\bullet$  are equivalent. More precisely, there is a square*

$$\begin{array}{ccc} MS(n) \times (\text{Tot}\mathcal{O}^\bullet)^n & \longrightarrow & \text{Tot}\mathcal{O}^\bullet \\ \downarrow & & \downarrow \\ \mathcal{D}_2(n) \times (\text{Tot}\mathcal{O}^\bullet)^n & \longrightarrow & \text{Tot}\mathcal{O}^\bullet \end{array}$$

that commutes for each  $n \geq 0$ .

## Outline of the paper.

- In Section 2 we recall the notions of cosimplicial spaces and totalizations. We also recall the notion of multiplicative operads.
- In Section 3 we recall the cacti operad  $MS$ , and we construct an explicit action on  $\text{Tot}\mathcal{O}^\bullet$ .
- In Section 4 we first recall the McClure-Smith operad  $\mathcal{D}_2$ . Then we explicit the details of its action on  $\text{Tot}\mathcal{O}^\bullet$ .
- In Section 5 we prove Theorem 1.1.

## 2 Cosimplicial spaces, totalizations, and multiplicative operads

This section recalls some basic notions. We also recall the fundamental construction  $\mathcal{O} \rightsquigarrow \mathcal{O}^\bullet$  due to McClure and Smith in [1].

Let us start with the notion of *cosimplicial spaces* and *totalizations*. Let  $\Delta$  be the category whose objects are ordered sets on the form  $[k] = \{0, \dots, k\}$ ,  $k \geq 0$ , and morphisms are non decreasing maps. A *cosimplicial space* is a covariant functor  $\mathcal{O}^\bullet: \Delta \rightarrow \text{Top}$  from  $\Delta$  to topological spaces. One can define a cosimplicial space as a sequence  $\mathcal{O}^\bullet = \{\mathcal{O}^k\}_{k \geq 0}$  of topological spaces equipped with maps  $d^i: \mathcal{O}^k \rightarrow \mathcal{O}^{k+1}$  (called *cofaces*) and  $s^j: \mathcal{O}^{k+1} \rightarrow \mathcal{O}^k$  (called *codegeneracies*) satisfying some identities (known as *cosimplicial relations*). One of the simplest examples of a cosimplicial space is the *standard cosimplicial space*  $\Delta^\bullet = \{\Delta^k\}_{k \geq 0}$ . The space  $\Delta^k$  is the standard geometric  $k$ -simplex. Throughout this paper it will be defined by

$$\Delta^k = \begin{cases} * & \text{if } k = 0 \\ \{(t_1, \dots, t_k) \in [-1, 1]^k : -1 \leq t_1 \leq \dots \leq t_k \leq 1\} & \text{if } k \geq 1. \end{cases}$$

The maps  $d^i: \Delta^k \rightarrow \Delta^{k+1}$  and  $s^j: \Delta^{k+1} \rightarrow \Delta^k$  are defined as follows.

- For  $0 \leq i \leq k+1$ , for  $t = (t_1, \dots, t_k) \in \Delta^k$ ,

$$d^i(t) = \begin{cases} (-1, t_1, \dots, t_k) & \text{if } i = 0 \\ (t_1, \dots, t_i, t_i, \dots, t_k) & \text{if } 1 \leq i \leq k \\ (t_1, \dots, t_k, 1) & \text{if } i = k+1. \end{cases}$$

- For  $1 \leq j \leq k+1$ ,  $t = (t_1, \dots, t_{k+1}) \in \Delta^{k+1}$ ,

$$s^j(t) = (t_1, \dots, t_{j-1}, t_{j+1}, \dots, t_{k+1}).$$

**Definition 2.1.** The totalization of a cosimplicial space  $\mathcal{O}^\bullet$ , denoted by  $\text{Tot}\mathcal{O}^\bullet$ , is the space of natural transformations from the standard cosimplicial space  $\Delta^\bullet$  to  $\mathcal{O}^\bullet$ . That is,

$$\text{Tot}\mathcal{O}^\bullet := \text{Nat}(\Delta^\bullet, \mathcal{O}^\bullet).$$

Let us define now the important notion of *multiplicative operads*. Roughly speaking, an element of an operad is an operation with many inputs and one output. To be more precise, we have the following definition.

**Definition 2.2.** An operad is a collection  $\mathcal{O} = \{\mathcal{O}(k)\}_{k \geq 0}$  of topological spaces together with an unit  $id \in \mathcal{O}(1)$  and insertion maps

$$\circ_i: \mathcal{O}(p) \times \mathcal{O}(q) \rightarrow \mathcal{O}(p+q-1), 1 \leq i \leq p,$$

such that for  $x \in \mathcal{O}(p)$ ,  $y \in \mathcal{O}(q)$ ,  $z \in \mathcal{O}(k)$ ,

$$\begin{aligned} (x \circ_i y) \circ_{j+q-1} z &= (x \circ_j z) \circ_i y \text{ for } 1 \leq i < j \leq p \\ x \circ_i (y \circ_j z) &= (x \circ_i y) \circ_{i+j-1} z \text{ for } 1 \leq i \leq p \text{ and } 1 \leq j \leq q \\ x \circ_i id &= id \circ_1 x = x. \end{aligned}$$

As said in the previous definition, all our operads are topological and nonsymmetric. The simplest example of operad is the *associative operad*  $\mathcal{A}s = \{\mathcal{A}s(k)\}_{k \geq 0}$ . Recall that each  $\mathcal{A}s(k)$  is the one point space  $*$ . In other words,  $\mathcal{A}s = \{*\}_{k \geq 0}$ .

**Definition 2.3.** A multiplicative operad is a topological nonsymmetric operad  $\mathcal{O}$  equipped with a map  $\mathcal{A}s \rightarrow \mathcal{O}$  of nonsymmetric operads from the associative operad  $\mathcal{A}s$  to  $\mathcal{O}$ .

The following remark gives an equivalent definition of a multiplicative operad.

**Remark 2.4.** A multiplicative structure on an operad  $\mathcal{O}$  is equivalent to having special operations  $e \in \mathcal{O}(0)$  and  $\mu \in \mathcal{O}(2)$  satisfying

$$\mu \circ_1 \mu = \mu \circ_2 \mu \quad \text{and} \quad \mu \circ_1 e = \mu \circ_2 e = id.$$

**Proposition 2.5.** [1, Section 10] To any multiplicative operad  $\mathcal{O}$ , one can associate a cosimplicial space  $\mathcal{O}^\bullet$ .

*Proof.* Let  $\mathcal{O} = \{\mathcal{O}(k)\}_{k \geq 0}$  be a multiplicative operad, and let  $e \in \mathcal{O}(0)$  and  $\mu \in \mathcal{O}(2)$  as in Remark 2.4. Define  $\mathcal{O}^k = \mathcal{O}(k)$ . Define also the cofaces morphisms  $d^i: \mathcal{O}^k \rightarrow \mathcal{O}^{k+1}$  and the codegeneracies morphisms  $s^j: \mathcal{O}^{k+1} \rightarrow \mathcal{O}^k$  by the following formulas.

- For  $0 \leq i \leq k+1$ ,  $x \in \mathcal{O}^k$ , define

$$d^i(x) = \begin{cases} \mu \circ_2 x & \text{if } i = 0 \\ x \circ_i \mu & \text{if } 1 \leq i \leq k \\ \mu \circ_1 x & \text{if } i = k+1. \end{cases}$$

- For  $1 \leq j \leq k+1$ ,  $y \in \mathcal{O}^{k+1}$ , define  $s^j(y) = y \circ_j e$

It is straightforward to check cosimplicial relations with  $d^i$  and  $s^j$  thus defined. □

In the rest of this paper  $\mathcal{O}$  is a multiplicative operad, and  $\mathcal{O}^\bullet$  is the associated cosimplicial space (as in Proposition 2.5).

### 3 The cacti operad $MS$ and its action on $\text{Tot}\mathcal{O}^\bullet$

In this section we define the cacti operad  $MS$  (see Definition 3.9) and show that it explicitly acts on  $\text{Tot}\mathcal{O}^\bullet$  (Theorem 3.11). In all this paper, for  $n \geq 0$ , the set  $\bar{n}$  is defined by  $\bar{n} = \{1, \dots, n\}$ .

We begin with the definition of the cacti operad  $MS$ . Let  $S^1$  be the unit circle viewed as the quotient of the interval  $I = [-1, 1]$  by the relation  $-1 \sim 1$ . Let  $\pi: [-1, 1] \rightarrow S^1$  denote the canonical surjection, and let  $*$  denote the base point of  $S^1$ . To define  $MS$  we need to define first a family of topological spaces

$$\mathcal{I} = \{\mathcal{I}_k(n) : n \geq 0 \text{ and } 0 \leq k \leq \infty\}.$$

Let  $n \geq 1$  be an integer. We start by defining the space  $\mathcal{I}_\infty(n)$ . Next we will define  $\mathcal{I}_k(n)$  as a subspace of  $\mathcal{I}_\infty(n)$ .

Let  $K = \{K_i = [x_i^K, x_{i+1}^K]\}_{i=0}^{p_K-1}$  be a family of closed subintervals of  $I$  satisfying the following two conditions:

(P<sub>1</sub>):  $p_K \geq n$ ,  $x_0^K = -1$ ,  $x_p^K = 1$  and the points  $x_0^K, \dots, x_p^K$  are pairwise distinct.

(P<sub>2</sub>): The intervals  $K_i$  define  $n$  1-manifolds  $I_1^K, \dots, I_n^K$  of disjoint interiors and with equal length. This means that each  $I_j^K$  is an union of some  $K_i$ .

We denote by  $\mathcal{P}_n$  the collection of such a family  $K$ . That is,

$$\mathcal{P}_n = \{K = \{K_i = [x_i^K, x_{i+1}^K]\}_{i=0}^{p_K-1} \mid K \text{ satisfies } (P_1) \text{ and } (P_2)\}. \quad (3.1)$$

The image of a  $n$ -tuple  $(I_1^K, \dots, I_n^K)$  (respectively the image of intervals  $K_i$ ) under the canonical surjection  $\pi$  will be denoted again by  $(I_1^K, \dots, I_n^K)$  (respectively by  $\{K_i\}$ ). The set  $\mathcal{I}_\infty(n)$  is then defined by

$$\mathcal{I}_\infty(n) = \{(I_1^K, \dots, I_n^K) \mid K \in \mathcal{P}_n\}.$$

From now and in the rest of this paper, we will denote an element  $x \in \mathcal{I}_\infty(n)$  by  $x = (I_1(x), \dots, I_n(x))$  or just by  $x = (I_1, \dots, I_n)$ . The family  $K = \{K_i = [x_i^K, x_{i+1}^K]\}_{i=0}^{p_K-1}$  will be sometimes just denoted by  $K = \{K_i = [x_i, x_{i+1}]\}_{i=0}^{p-1}$ .

Let us equip now the set  $\mathcal{I}_\infty(n)$  with the following topology. Two elements

$$x = (I_1(x), \dots, I_n(x)) \quad \text{and} \quad y = (I_1(y), \dots, I_n(y))$$

of  $\mathcal{I}_\infty(n)$  are said to be closed if the 1-manifolds  $I_i(x)$  and  $I_i(y)$  are closed (in the sense that we have the inequality  $\text{length}(I_i(x) \setminus \overset{\circ}{I}_i(y)) < \epsilon$  for some  $\epsilon > 0$  too small) to each other for all  $i$ . Notice that  $\mathcal{I}_\infty(1)$  is the one point space. In order to define the space  $\mathcal{I}_k(n)$  (for  $k \geq 0$  be an integer), recall first the notion of the *complexity* of a map.

**Definition 3.1.** Let  $T$  be a finite totally ordered set,  $n$  be an integer, and  $f: T \longrightarrow \bar{n} = \{1, \dots, n\}$  be a map. The complexity of  $f$ , denoted by  $\text{cplx}(f)$ , is defined as follows.

- If  $n = 0$  or  $n = 1$  then  $\text{cplx}(f) = 0$ .
- If  $n = 2$ , let  $\sim$  be the equivalence relation on  $T$  generated by

$$a \sim b \text{ if } a \text{ is adjacent to } b \text{ and } f(a) = f(b).$$

The complexity of  $f$  is equal to the number of equivalence classes minus 1.

- If  $n > 2$ , let  $f_{ij}: f^{-1}(\{i, j\}) \longrightarrow \{i, j\}$  denote the restriction of  $f$  on  $f^{-1}(\{i, j\})$ . The complexity of  $f$  is equal to the maximum of complexities of the restrictions  $f_{ij}$  as  $\{i, j\}$  ranges over the two-element subsets of  $\bar{n}$ . That is,

$$\text{cplx}(f) = \text{Max}_{1 \leq i < j \leq n} (\text{cplx}(f_{ij})). \quad (3.2)$$

Note that a map  $f: T \longrightarrow \{1, \dots, n\}$  can be viewed as a word of length  $|T|$  on the alphabet  $\{1, \dots, n\}$  (here  $|T|$  denotes the cardinal of  $T$ ). The *length of the alphabet*  $\{1, \dots, n\}$  is defined to be its cardinal.

**Example 3.2.** (a) If  $|T| = 5$ ,  $n = 2$  and  $f$  is defined by the word  $f = 12212$  then  $\text{cplx}(f) = 3$ .

(b) Assume that  $|T| = 8$ ,  $n = 3$  and  $f = 31232113$ , and consider the following tabular

map	$f_{12} = 12211$	$f_{13} = 313113$	$f_{23} = 32323$
complexity	$\text{cplx}(f_{12}) = 2$	$\text{cplx}(f_{13}) = 4$	$\text{cplx}(f_{23}) = 4$

By this tabular and by (3.2), we deduce that  $\text{cplx}(f) = 4$ .

Let  $x = (I_1, \dots, I_n) \in \mathcal{I}_\infty(n)$  defined by a partition  $K = \{K_i\}_{i=0}^{p-1} \in \mathcal{P}_n$ . Let  $T_x$  be the set defined by  $T_x = \{0, 1, \dots, p-1\}$ . For each  $k \in T_x$ , it is clear that there exists a unique  $i_k \in \{1, \dots, n\}$  such that  $K_k \subseteq I_{i_k}$  (this comes from the condition  $(P_2)$  above). This defines a map

$$f_x: T_x \longrightarrow \{1, \dots, n\} \quad (3.3)$$

**Definition 3.3.** The complexity of an element  $x \in \mathcal{I}_\infty(n)$ , denoted by  $\text{cplx}(x)$ , is defined to be the complexity of the map  $f_x$ . That is,

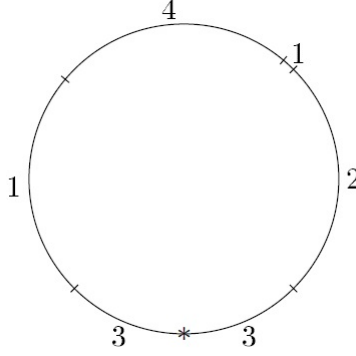
$$\text{cplx}(x) = \text{cplx}(f_x).$$

We are now ready to define  $\mathcal{I}_k(n)$ .

**Definition 3.4.** For  $k \geq 0$ , the space  $\mathcal{I}_k(n)$  is the subspace of  $\mathcal{I}_\infty(n)$  defined by

$$\mathcal{I}_k(n) = \{x \in \mathcal{I}_\infty(n) \mid \text{cplx}(x) \leq k\}.$$

The following figure is an element of  $\mathcal{I}_2(4)$ .



**Remark 3.5.** The space  $\mathcal{I}_2(n)$  is a finite regular CW-complex with one cell for each  $f_x(T_x)$  (see (3.3) above for the definition of  $f_x$ ). A cell labeled by some  $f_x(T_x)$  is homeomorphic to  $\prod_{i=1}^n \Delta^{|f_x^{-1}(i)|-1}$ . For instance, the figure above is an element of a cell homeomorphic to  $\Delta^1 \times \Delta^1$ , which is labeled by 314123.

Now we want to define what so called *cactus with  $n$  lobes*. Let  $x = (I_1, \dots, I_n) \in \mathcal{I}_2(n)$  defined by a family of closed intervals  $K = \{K_{i1}, \dots, K_{il_i}\}_{i=1}^q \in \mathcal{P}_n$ . Assume that each  $I_i$  is on the form  $I_i = \cup_{r=1}^{l_i} K_{ir}$ , and define on  $S^1$  the equivalence relation  $\sim_i$  (for  $1 \leq i \leq n$ ) generated by

$$(t_1 \sim_i t_2) \text{ if and only if } (t_1, t_2 \in K_{jr} \text{ with } jr \notin \{i1, \dots, il_i\}).$$

It is easy to see that the quotient of  $S^1$  by this equivalence is homeomorphic to  $S^1$ . Let us denote by  $\pi_i$  the canonical surjection.

$$\pi_i: S^1 \longrightarrow S^1 / \sim_i \cong S^1.$$

We thus construct a map

$$c(x): S^1 \longrightarrow (S^1)^n$$

defined by  $c(x) = (\pi_1, \dots, \pi_n)$ , and called the *cactus map*. Its image is called the *cactus with  $n$  lobes* associated to  $x \in \mathcal{I}_2(n)$ . Let  $\text{Coend}_{S^1}$  denote the coendomorphism operad on  $S^1$ . Then there is an embedding

$$\bar{\tau}_n: \mathcal{I}_2(n) \longrightarrow \text{Coend}_{S^1}(n)$$

defined by

$$\bar{\tau}(x) = c(x).$$

**Remark 3.6.** The collection of spaces  $\{\bar{\tau}_n(\mathcal{I}_2(n))\}_{n \geq 0}$  is not a suboperad of  $\text{Coend}_{S^1}(\bullet)$ . Indeed, let  $x$  be an element of  $\mathcal{I}_2(2)$  labeled by 212. Then  $c(x) = (\pi_1, \pi_2) \in \text{Coend}_{S^1}(2)$ . Using now the operad structure of  $\text{Coend}_{S^1}$ , we get

$$c(x) \circ c(x) = (\pi_1, \pi_1 \circ \pi_2, \pi_2 \circ \pi_2) = (\pi_1, \text{constant map}, \pi_2 \circ \pi_2),$$

and it is impossible to find an element  $z \in \mathcal{F}_2(3)$  such that  $c(z) = (\pi_1, \text{constant map}, \pi_2 \circ \pi_2)$ .

Since the collection  $\{\bar{\tau}_n(\mathcal{F}_2(n))\}_{n \geq 0}$  is not far to be an operad, to get the right one, we introduce the space  $\text{Mon}(I, \partial I)$  defined as follows. Let  $\partial I = \{-1, 1\}$  denotes the boundary of  $I$ . Let  $\varphi: I \rightarrow I$  be a weakly monotone map such that its restriction on  $\partial I$  coincides with the identity map  $\text{id}_{\partial I}$ . Then the map  $\varphi$  passes to the quotient and gives a map  $\tilde{\varphi}: S^1 \rightarrow S^1$ , which is a typical element of  $\text{Mon}(I, \partial I)$ .

**Remark 3.7.** The space  $\text{Mon}(I, \partial I)$  is homeomorphic to the totalization  $\text{Tot} \Delta^\bullet \simeq *$ . The homeomorphism  $\text{Mon}(I, \partial I) \xrightarrow{\cong} \text{Tot} \Delta^\bullet$  sends  $\tilde{\varphi} \in \text{Mon}(I, \partial I)$  to  $(t_1, \dots, t_k) \mapsto (\varphi(t_1), \dots, \varphi(t_k))$ .

Considering the embedding

$$\tau_n: \mathcal{I}_2(n) \times \text{Mon}(I, \partial I) \rightarrow \text{Coend}_{S^1}(n)$$

defined by

$$\tau_n(x, \tilde{\varphi}) = c(x) \circ \tilde{\varphi}: S^1 \rightarrow (S^1)^n,$$

we have the following proposition.

**Proposition 3.8.** The collection  $\{\text{im}(\tau_n)\}_{n \geq 0} = \{\tau_n(\mathcal{I}_2(n) \times \text{Mon}(I, \partial I))\}_{n \geq 0}$  is a suboperad of  $\text{Coend}_{S^1}$ .

*Proof.* The proof follows immediately from [2, Proposition 4.5].  $\square$

Let  $MS = \{MS(n)\}_{n \geq 0}$  be the collection of topological spaces defined by

$$MS(n) = \begin{cases} \mathcal{I}_2(n) \times \text{Mon}(I, \partial I) & \text{if } n \geq 1 \\ * & \text{if } n = 0. \end{cases} \quad (3.4)$$

By transferring the operad structure of  $\{\text{im}(\tau_n)\}_{n \geq 0}$  (given by Proposition 3.8) on  $MS$  via embeddings  $\tau_n$ , we endow  $MS$  with an operad structure.

**Definition 3.9.** The operad  $MS$  is called the *cacti operad*.

**Proposition 3.10.** There exists a weak equivalence  $(\phi, \text{id}): S^1 \xrightarrow{\sim} MS(2)$ .

*Proof.* Define  $\phi: S^1 \rightarrow \mathcal{I}_2(2)$  by

$$\phi(\tau) = \begin{cases} (I_1 = K_0 \cup K_2, I_2 = K_1), K_0 = [-1, \tau], K_1 = [\tau, 1 + \tau], K_2 = [1 + \tau, 1] & \text{if } -1 < \tau < 0 \\ (I_1 = K_1, I_2 = K_0 \cup K_2), K_0 = [-1, \tau - 1], K_1 = [\tau - 1, \tau], K_2 = [\tau, 1] & \text{if } 0 < \tau < 1 \\ (I_1 = K_0, I_2 = K_1), K_0 = [-1, 0], K_1 = [0, 1] & \text{if } \tau = 0 \\ (I_1 = K_1, I_2 = K_0), K_0 = [-1, 0], K_1 = [0, 1] & \text{if } \tau = \pm 1 \end{cases} \quad (3.5)$$

It is not difficult to see that  $\phi$  is a homeomorphism. Therefore the map

$$(\phi, \text{id}): S^1 \rightarrow MS(2) = \mathcal{I}_2(2) \times \text{Mon}(I, \partial I), \quad (3.6)$$

where  $\text{id}$  is the map that sends each point of  $S^1$  to the identity map  $\text{id}_{S^1}: S^1 \rightarrow S^1$ , is a weak equivalence since the space  $\text{Mon}(I, \partial I)$  is contractible by Remark 3.7.  $\square$

The following theorem is originally due to P. Salvatore in [2]. He gives a nice geometric proof to it. Here we furnish another proof, which is more combinatorial. We provide in fact explicit formulas of the action of  $MS$  on  $\text{Tot} \mathcal{O}^\bullet$ .

**Theorem 3.11.** [2, Theorem 5.4] *Let  $\mathcal{O}^\bullet$  be a cosimplicial space defined by a multiplicative operad  $\mathcal{O}$ . Then the cacti operad  $MS$  acts on the totalization  $\text{Tot}\mathcal{O}^\bullet$ .*

*Proof.* Let  $(x, \tilde{\varphi}) \in MS(n) = \mathcal{I}_2(n) \times \text{Mon}(I, \partial I)$ , let  $a_i^\bullet \in \text{Tot}\mathcal{O}^\bullet$ ,  $1 \leq i \leq n$ . Our aim is to construct from these data a family

$$\theta_n((x, \tilde{\varphi}), (a_1^\bullet, \dots, a_n^\bullet))_k : \Delta^k \longrightarrow \mathcal{O}^k, k \geq 0, \quad (3.7)$$

of maps that commute with cofaces and codegeneracies.

Let  $t = (-1 = t_0 \leq t_1 \leq \dots \leq t_k \leq t_{k+1} = 1)$  be an element of  $\Delta^k$ . Define a family of closed intervals  $\{J_j\}_{j=0}^k$  by setting  $J_j = [\varphi(t_j), \varphi(t_{j+1})]$ . Assume that  $x \in \mathcal{I}_2(n)$  is defined by a family of  $p$  closed intervals  $K = \{[x_i, x_{i+1}]\}_{i=0}^{p-1}$ , and consider the set

$$E = \{\varphi(t_j) \mid 0 \leq j \leq k+1\} \cup \{x_i \mid 0 \leq i \leq p\}.$$

Define now a family  $\{L_l = [a_l, b_l]\}_{l=0}^m$  of closed subintervals of  $I$  as follows.

- each  $a_l$  or  $b_l$  belongs to  $E$ ;
- the interior of each  $L_l$  contains no element of  $E$ ;
- $\cup_{l=0}^m L_l = I$  and

$$m = k + p - 1. \quad (3.8)$$

**(a) Assume that for all  $i$  and for all  $j$ ,  $x_i \neq \varphi(t_j)$**

In this case there exists, for each  $l \in \{0, \dots, m\}$ , an unique element  $i_l \in \{1, \dots, n\}$  and an unique element  $j_l \in \{0, \dots, k\}$  such that  $L_l \subseteq I_{i_l}$  and  $L_l \subseteq J_{j_l}$ . This gives two maps

$$[k] \xleftarrow{h} [m] \xrightarrow{f} \bar{n} \quad (3.9)$$

defined by

$$h(l) = j_l \quad \text{and} \quad f(l) = i_l.$$

It is easy to see that the map  $h$  is a morphism in the simplicial category  $\Delta$ . It is also easy to see that  $f$  is a surjective map, and its complexity is less than or equal to 2 (this is because we have taken  $x$  in  $\mathcal{I}_2(n)$ , and by Definition 3.4 we have  $\text{cplx}(x) \leq 2$ ).

Let  $i \in \bar{n}$ . We want to explicitly construct an element  $y_i \in \Delta^{f^{-1}(i)} = \Delta^{|f^{-1}(i)|-1}$ . Let us set

$$I_i = \cup_{j=0}^{l_i} K_{i_j}, K_{i_j} = [x_{i_j}, x_{i_{j+1}}] \subseteq [-1, 1], \text{ and } x_{i_{l_i+1}} = 1.$$

Define by induction a family  $\{g_{i_q} : [-1, 1] \longrightarrow [-1, 1]\}_{q=0}^{l_i+1}$  of maps as follows.

$$g_{i_0}(z) = \begin{cases} -1 & \text{if } z \in [-1, x_{i_0}] \\ z - (x_{i_0} + 1) & \text{if } z > x_{i_0}, \end{cases} \quad (3.10)$$

and for  $0 \leq q \leq l_i$ ,

$$g_{i_{q+1}}(z) = \begin{cases} z & \text{if } z \in [-1, g_{i_q} \circ \dots \circ g_{i_0}(x_{i_{q+1}})] \\ g_{i_q} \circ \dots \circ g_{i_0}(x_{i_{q+1}}) & \text{if } z \in [g_{i_q} \circ \dots \circ g_{i_0}(x_{i_{q+1}}), g_{i_q} \circ \dots \circ g_{i_0}(x_{i_{q+1}})] \\ z - (x_{i_{q+1}} - x_{i_q}) & \text{if } z > g_{i_q} \circ \dots \circ g_{i_0}(x_{i_{q+1}}) \end{cases} \quad (3.11)$$

Intuitively, the map  $g_{i_0}$  contracts the interval  $[-1, x_{i_0}]$  to  $-1$ , and moves other points by the translation of vector  $-(x_{i_0} + 1)$ , the map  $g_{i_1}$  contracts the interval  $[g_{i_0}(x_{i_0+1}), g_{i_0}(x_{i_1})]$  to  $g_{i_0}(x_{i_0+1})$ , and moves other points by the translation of vector  $-(x_{i_1} - x_{i_0+1})$ , and so on. At the end of this process, we obtain an interval of length  $\frac{2}{n}$ . More precisely, if we define  $g$  to be the composite

$$g = g_{i_{l_i}+1} \circ g_{i_{l_i}} \circ \cdots \circ g_{i_1} \circ g_{i_0}, \quad (3.12)$$

then

$$g([-1, 1]) = [-1, \frac{2-n}{n}].$$

Define also a map  $\alpha: [-1, \frac{2-n}{n}] \rightarrow [-1, 1]$  by

$$\alpha(z) = nz + n - 1. \quad (3.13)$$

Notice the map  $\alpha$  fixes  $-1$ , and sends  $\frac{2-n}{n}$  to  $1$ . In fact  $\alpha$  allows to rescale the interval  $[-1, \frac{2-n}{n}]$ . Consider now the map  $\tilde{g}: [-1, 1] \rightarrow [-1, 1]$  defined by

$$\tilde{g} = \alpha \circ g.$$

For  $j \in \{0, \dots, l_i\}$ , if  $t_{i_j}^1, \dots, t_{i_j}^{v_j}$  denote elements of the set  $\{\varphi(t_1), \dots, \varphi(t_k)\}$  that belong to  $K_{i_j}$ , then  $y_i \in \Delta^{|f^{-1}(i)|-1}$  is defined by

$$y_i = (\tilde{g}(t_{i_0}^1), \dots, \tilde{g}(t_{i_0}^{v_0}), \tilde{g}(x_{i_0+1})), \tilde{g}(t_{i_1}^1), \dots, \tilde{g}(t_{i_1}^{v_1}), \dots, \tilde{g}(x_{i_{l_i-1}+1}), \tilde{g}(t_{i_{l_i}}^1), \dots, \tilde{g}(t_{i_{l_i}}^{v_{l_i}})). \quad (3.14)$$

An illustration for  $y_i$  is given in the first part of Example 3.12.

Let us construct now by induction on  $n$  the operation  $\theta_n(((x, \tilde{\varphi}), (a_1^\bullet, \dots, a_n^\bullet)))_k(t) \in \mathcal{O}(k)$ . The map  $f$  will be thought as a word of length  $m+1$  on the alphabet  $\{1, \dots, n\}$ . If  $W$  is a word on an alphabet of length  $*$ , we will write  $\theta_*(W)$  for the associated operation. For instance, the operation  $\theta_n(((x, \tilde{\varphi}), (a_1^\bullet, \dots, a_n^\bullet)))_k(t)$  will be sometimes denoted by  $\theta_n(f)$ .

If  $n = 1$  then  $c(x) = id_{S^1}$ . Define  $\theta_1(((x, \tilde{\varphi}), a_1^\bullet))_k(t) \in \mathcal{O}(k)$  by

$$\theta_1(((x, \tilde{\varphi}), a_1^\bullet))_k(t) = a_1^k(t)$$

If  $n = 2$ , let  $i, j$  be two distinct elements inside  $\{1, 2\}$ . Since the complexity of  $f$  is less than or equal to 2, there are two possibilities for writing the word  $f$ .

- Assume that the map  $f: [m] \rightarrow \{1, 2\}$  is on the form  $f = \underbrace{i \cdots i}_{r+1} \underbrace{j \cdots j}_{s+1}$ .

This implies that we have exactly two closed intervals  $K_0, K_1$  defining  $x = (I_1, I_2) \in \mathcal{I}_2(2)$ , and therefore,  $p = 2$  (recall that  $p$  is the number of intervals  $K_i$  defining  $x \in \mathcal{I}_2(2)$ ). We claim that  $r + s = k$ . Indeed, since the length of the word  $f$  is equal to  $m+1$ , it follows that

$$\begin{aligned} (r+1) + (s+1) &= m+1 \\ &= k+p \quad \text{since } m = k+p-1 \text{ by (3.8) above} \\ &= k+2 \quad \text{since } p = 2. \end{aligned}$$

Let  $\mu \in \mathcal{O}(2)$  denote the multiplication. Define  $\theta_2(((x, \tilde{\varphi}), (a_1^\bullet, a_2^\bullet)))_k(t) \in \mathcal{O}(r+s) = \mathcal{O}(k)$  by the formula

$$\theta_2(((x, \tilde{\varphi}), (a_1^\bullet, a_2^\bullet)))_k(t) = \theta_2(i \cdots i j \cdots j) = \mu(a_i^r(y_i), a_j^s(y_j)). \quad (3.15)$$



- Now we assume that the word  $f$  is on the form  $f = \underbrace{i \cdots i}_{r_1} \underbrace{j \cdots j}_{s+1} \underbrace{i \cdots i}_{r_2}$ .

This implies that  $p = 3$ . Like before, we can check that  $r_1 + r_2 - 1 + s - 1 = k$ . Define the operation  $\theta_2((x, \tilde{\varphi}), (a_1^\bullet, a_2^\bullet))_k(t) \in \mathcal{O}(r_1 + r_2 - 1 + s - 1) = \mathcal{O}(k)$  by

$$\theta_2((x, \tilde{\varphi}), (a_1^\bullet, a_2^\bullet))_k(t) = \theta_2(i \cdots i j \cdots j i \cdots i) = a_i^{r_1+r_2-1}(y_i) \circ_{r_1} a_j^s(y_j). \quad (3.16)$$

Let  $n \geq 3$ . For  $i \leq n - 1$ , the operation  $\theta_i((x, \tilde{\varphi}), (a_1^\bullet, \dots, a_i^\bullet))_k$  will be denoted just by  $\theta_*(-)$ . Assume that  $\theta_*(W)$  is constructed for each word  $W$  on an alphabet of length  $* \leq n - 1$ . We want to construct  $\theta_n(f)$ . Set  $f(0) = i_0$  and define the integer

$$m_0 = \text{Max}\{j \in [m] \mid f(j) = i_0\}.$$

There are two possibilities depending of the fact that the word  $f$  ends by the letter  $i_0$  or not.

- If  $m_0 = m$  then the map  $f$  is on the form

$$f = \underbrace{i_0 \cdots i_0}_{r_1} b_{11} \cdots b_{1s_1} \underbrace{i_0 \cdots i_0}_{r_2} b_{21} \cdots b_{2s_2} \cdots \underbrace{i_0 \cdots i_0}_{r_q} b_{q1} \cdots b_{qs_q} \underbrace{i_0 \cdots i_0}_{r_{q+1}},$$

with  $b_{js} \neq i_0$  for all  $j$  and for all  $s$ , and with  $r_1 + \cdots + r_{q+1}$  copies of  $i_0$ . Let us set

$$u = \left( \sum_{i=1}^{q+1} r_i \right) - 1 \quad \text{and} \quad v = \sum_{i=1}^q r_i.$$

Define  $\theta_n(f)$  by the formula

$$\theta_n(f) = (((a_{i_0}^u(y_{i_0}) \circ_v \theta_*(b_{q1} \cdots b_{qs_q})) \circ_{v-r_q} \cdots) \circ_{r_1} \theta_*(b_{11} \cdots b_{1s_1})). \quad (3.17)$$

A perfect illustration for this formula is given by Example 3.13 below.

- If  $m_0 < m$  then the map  $f$  is on the form

$$f = i_0 \cdots i_0 b_1 \cdots b_w i_0 \cdots i_0 c_1 \cdots c_s$$

with  $c_j \neq i_0$  for all  $j$ . Let  $r$  be the number of letters (in the alphabet  $\{1, \dots, n\}$ ) appearing in the word  $f(\{0, \dots, m_0\}) = i_0 \cdots i_0 b_1 \cdots b_w i_0 \cdots i_0$ . Then, since each letter of the word  $b_1 \cdots b_w$  does not appear in the word  $c_1 \cdots c_s$  (because the complexity of  $f$  is less than or equal to 2), there is exactly  $n - r$  letters appearing in the word  $c_1 \cdots c_s$ . Define

$$\theta_n(f) = \mu(\theta_r(i_0 \cdots i_0 b_1 \cdots b_w i_0 \cdots i_0), \theta_{n-r}(c_1 \cdots c_s)), \quad (3.18)$$

**(b) Now we assume that there exists some integers  $i$  and  $j$  such that  $x_i = \varphi(t_j)$**

In this case, there is a finite number of possibilities (depending of the fact that we consider the interval  $[\varphi(t_j), x_i]$  or  $[x_i, \varphi(t_j)]$  in the family  $\{L_l\}_{l=0}^m$ ) to define the word  $f$ . It is not difficult to show, by using the naturality of the maps  $a_r^\bullet: \Delta^\bullet \rightarrow \mathcal{O}^\bullet$  and the fact that  $\mu(\mu, id) = \mu(id, \mu)$  (we have this equality because  $\mathcal{O}$  is a multiplicative operad), that all these possibilities lead to the same element  $\theta_n((x, \tilde{\varphi}), (a_1^\bullet, \dots, a_n^\bullet))_k(t) \in \mathcal{O}(k)$ . A good illustration of that is given by the second part of Example 3.12 below.

We can check that the maps  $\theta_n((x, \tilde{\varphi}), (a_1^\bullet, \dots, a_n^\bullet))_k: \Delta^k \rightarrow \mathcal{O}^k$  thus defined are continuous and commute with cofaces and codegeneracies. The continuity comes essentially from the fact that  $\mu(\mu, id) = \mu(id, \mu)$ . The continuity of the map  $\theta_n: MS(n) \times (\text{Tot } \mathcal{O}^\bullet)^n \rightarrow \text{Tot } \mathcal{O}^\bullet$  also comes from the same fact.  $\square$

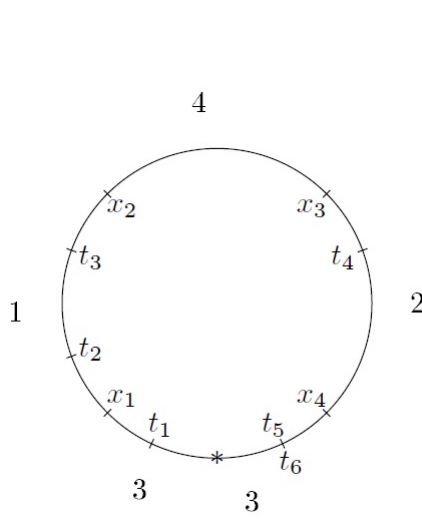


Figure a

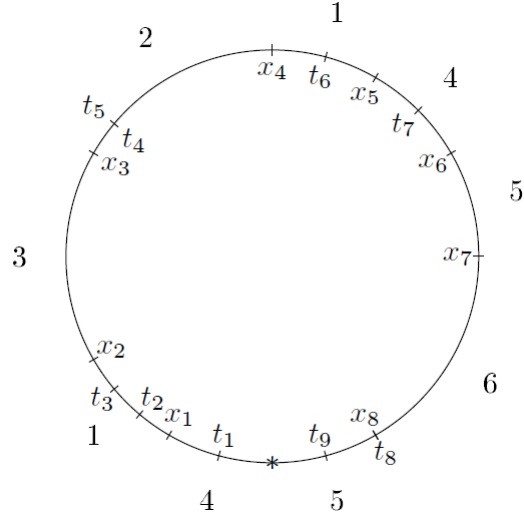


Figure b

**Example 3.12.** (a) Let  $n = 4$ ,  $x \in \mathcal{I}_2(4)$  and  $t = (t_1, \dots, t_6) \in \Delta^6$  (see Figure a). Assume that  $\tilde{\varphi} = id_{S^1}$ . Then  $k = 6$ ,  $p = 5$ ,  $m = k + p - 1 = 10$  and the map  $f: [10] \rightarrow \{1, 2, 3, 4\}$  is defined by the word  $f = 33111422333$ . Now let us explicitly define  $y_1 \in \Delta^{|f^{-1}(1)|} = \Delta^2$ . First of all, we have  $I_1 = [x_1, x_2]$ . So the map  $g$  (see (3.12)) is just equal to  $g_{i_0}$  (notice that here  $i_0 = 1$ ), and by the definition of this latter map (see (3.10)) we have  $g_{i_0}(z) = z - (x_1 + 1)$  for each  $z \in [x_1, x_2]$ . On the other hand, the map  $\alpha$  here is defined by  $\alpha(z) = 4z + 3$  (see (3.13)). Therefore the image of each  $z \in [x_1, x_2]$  under the composite  $\tilde{g} = \alpha \circ g_{i_0}$  gives  $4z - 4x_1 - 1$ . Hence,

$$y_1 = (4t_2 - 4x_1 - 1, 4t_3 - 4x_1 - 1) \in \Delta^2$$

A similar computation gives  $y_2 = 4t_4 - 4x_3 - 1 \in \Delta^1$ . For  $y_3$  we use the formula (3.14), and we obtain

$$y_3 = (4t_1 + 3, 4x_1 + 3, 4t_5 - 4x_4 + 4x_1 + 3, 4t_6 - 4x_4 + 4x_1 + 3) \in \Delta^4.$$

Now we can define  $\theta_4(f)$ .

$$\begin{aligned} \theta_4(f) &= a_3^4(y_3) \circ_2 \theta_3(111244) \text{ by (3.17)} \\ &= a_3^4(y_3) \circ_2 \mu(a_1^2(y_1), \theta_2(244)) \text{ by (3.18)} \\ &= a_3^4(y_3) \circ_2 \mu(a_1^2(y_1), \mu(a_4^0(*), a_2^1(y_2))) \in \mathcal{O}(6) \text{ by (3.15)}. \end{aligned}$$

(b) Let  $n = 6$ ,  $x \in \mathcal{I}_2(6)$  and  $t = (t_1, \dots, t_9) \in \Delta^9$  (see Figure b above). Assume that  $\tilde{\varphi} = id_{S^1}$ . Then  $k = 9$ ,  $p = 9$  and  $m = k + p - 1 = 17$ . Since  $t_8 = x_8$ , it follows that there are two possibilities to define the map  $f: [17] \rightarrow \{1, 2, 3, 4, 5, 6\}$ .

If  $f = 441113222114456655$  then we have (by applying formulas (3.18), (3.17), (3.15) and (3.16))

$$\theta_6(f) = \mu(a_4^3(y_4) \circ_2 (a_1^4(y_1) \circ_3 \mu(a_3^0(*), a_2^2(y_2))), a_5^2(y_5) \circ_1 a_6^1(y_6)) \in \mathcal{O}(9).$$

Here  $y_5 = (-1 \leq y_{51} \leq y_{52} \leq 1)$  is an element of  $\Delta^2$ . Let us denote it by  $y_5^2$ .

If  $f = 441113222114456555$  then we have (again by formulas (3.18), (3.17), (3.15) and (3.16))

$$\theta_6(f) = \mu(a_4^3(y_4) \circ_2 (a_1^4(y_1) \circ_3 \mu(a_3^0(*), a_2^2(y_2))), a_5^3(y_5) \circ_1 a_6^0(*)) \in \mathcal{O}(9).$$

Here  $y_5 = (-1 \leq y_{50} \leq y_{51} \leq y_{52} \leq 1)$  is an element of  $\Delta^3$  with  $y_{50} = y_{51}$ . Let us denote it by  $y_5^3$ .

To check that these two possibilities lead to the same operation in  $\mathcal{O}(9)$ , it suffices to check that

$$a_5^3(y_5^3) \circ_1 a_6^0(*) = a_5^2(y_5^2) \circ_1 a_6^1(y_6).$$

Here we go

$$\begin{aligned}
a_5^3(y_5^3) \circ_1 a_6^0(*) &= a_5^3(y_{50}, y_{51}, y_{52}) \circ_1 a_6^0(*) \text{ because } y_5^3 = (y_{50}, y_{51}, y_{52}) \text{ with } y_{50} = y_{51} \\
&= a_5^3(d^1(y_{51}, y_{52})) \circ_1 a_6^0(*) \text{ because } (y_{51}, y_{51}, y_{52}) = d^1(y_{51}, y_{52}), d^1: \Delta^2 \longrightarrow \Delta^3 \\
&= (d^1(a_5^2(y_{51}, y_{52}))) \circ_1 a_6^0(*) \text{ by the naturality of } a_5^\bullet \\
&= (a_5^2(y_{51}, y_{52}) \circ_1 \mu) \circ_1 a_6^0(*) \text{ by the definition of the coface map } d^1: \mathcal{O}^2 \longrightarrow \mathcal{O}^3 \\
&= a_5^2(y_{51}, y_{52}) \circ_1 (\mu \circ_1 a_6^0(*)) \text{ by the associativity in } \mathcal{O} \\
&= a_5^2(y_{51}, y_{52}) \circ_1 (d^1(a_6^0(*))) \text{ by the definition of the coface map } d^1: \mathcal{O}^0 \longrightarrow \mathcal{O}^1 \\
&= a_5^2(y_{51}, y_{52}) \circ_1 a_6^1(d^1(*)) \text{ by the naturality of } a_6^\bullet \\
&= a_5^2(y_5^2) \circ_1 a_6^1(y_6).
\end{aligned}$$

**Example 3.13.** In this example, we are in the case (a) of the proof of Theorem 3.11. Take  $t = (t_1, \dots, t_{14}) \in \Delta^{14}$  such that  $t_i \neq t_j$  whenever  $i \neq j$ , and take  $f = 111224422111133355511$ . The operation  $\theta_5(f) \in \mathcal{O}(14)$  is then defined by

$$\begin{aligned}
\theta_5(f) &= (a_1^8(y_1) \circ_7 \theta_2(3335555)) \circ_3 \theta_2(224422) \text{ by (3.17)} \\
&= (a_1^8(y_1) \circ_7 \mu(a_3^2(y_3), a_5^3(y_5))) \circ_3 (a_2^3(y_2) \circ_2 a_4^1(y_4)) \text{ by (3.15) and (3.16)}.
\end{aligned}$$

## 4 McClure-Smith operad $\mathcal{D}_2$ and its action on $\text{Tot}\mathcal{O}^\bullet$

Here we recall the construction of the operad  $\mathcal{D}_2$ . We also give the details of its action on  $\text{Tot}\mathcal{O}^\bullet$  (notice that this action was built by McClure and Smith in [1]). We will write  $\Delta_+$  for the category  $\Delta \cup \{\emptyset\}$ , and a covariant functor  $X^\bullet: \Delta_+ \longrightarrow \text{Top}$  will be called an *augmented cosimplicial space*. In all this section, the standard cosimplicial space  $\Delta^\bullet$  will be viewed as an augmented cosimplicial space with  $\Delta^\emptyset = \emptyset$ .

Let us begin with the definition of the augmented cosimplicial space

$$\Xi_n^2(X^\bullet, \dots, X^\bullet): \Delta_+ \longrightarrow \text{Top}$$

in which we have  $n$  copies of  $X^\bullet$ .

Let  $\bar{n}$  as in the previous section. Define  $Q_n$  to be the category whose objects are pairs  $(T, f)$ , where  $T$  is an object of the category  $\Delta_+$  and  $f: T \longrightarrow \bar{n}$  is a morphism in sets. A morphism from  $(T, f)$  to  $(T', f')$  consists of a morphism  $g: T \longrightarrow T'$  in  $\Delta_+$  such that  $f = f'g$ . Define also  $Q_n^2$  to be the full subcategory of  $Q_n$  consisting of pairs  $(T, f)$  such that  $\text{cplx}(f) \leq 2$ . Consider now the diagram

$$\begin{array}{ccccc}
Q_n^2 & \xrightarrow{\rho_2} & Q_n & \xrightarrow{\psi_n} & \text{Top}^n & \xrightarrow{\Pi_n} & \text{Top} \\
\downarrow \phi_n & & & & & & \\
\Delta_+ & & & & & & 
\end{array}$$

in which

- $\rho_2$  is the inclusion functor,
- $\psi_n$  is defined to be  $\psi_n(T, f) = (X^{f^{-1}(1)}, \dots, X^{f^{-1}(n)})$ ,
- $\Pi_n$  is the product functor and
- $\phi_n$  is the projection on the first component.

The covariant functor  $\Xi_n^2(X^\bullet, \dots, X^\bullet): \Delta_+ \longrightarrow \text{Top}$  is defined to be the left Kan extension of the composite  $\Pi_n \circ \psi_n \circ \rho_2$  along  $\phi_n$ . By the definition of a left Kan extension, the functor  $\Xi_n^2(X^\bullet, \dots, X^\bullet)$  is explicitly defined as follows.

Let  $S$  be an object of the category  $\Delta_+$ . We want to define the space  $\Xi_n^2(X^\bullet, \dots, X^\bullet)(S)$  associated to  $S$ . First we define the category  $A_{nS}$  whose objects are triples  $(h, T, f)$  where  $T$  is an object in  $\Delta_+$ ,  $h: T \rightarrow S$  is a morphism in  $\Delta_+$ , and  $(T, f)$  is an object in  $Q_n^2$ . A morphism from  $(h, T, f)$  to  $(h', T', f')$  consists of a morphism  $g: T \rightarrow T'$  such that  $h = h'g$  and  $f = f'g$ . We will sometimes write  $S \xleftarrow{h} T \xrightarrow{f} \bar{n}$  for an object  $(h, T, f)$  of the category  $A_{nS}$ . Next we define the functor  $p_n: A_{nS} \rightarrow Q_n^2$  by  $p_n(h, T, f) = (T, f)$ , and we consider the  $A_{nS}$ -diagram

$$F_{nS} = \prod_n \circ \psi_n \circ \rho_2 \circ p_n: A_{nS} \rightarrow \text{Top}.$$

The space  $\Xi_n^2(X^\bullet, \dots, X^\bullet)(S)$  is then explicitly defined to be the colimit of  $F_{nS}$ . That is,

$$\Xi_n^2(X^\bullet, \dots, X^\bullet)(S) = \text{colim}_{A_{nS}} F_{nS}. \quad (4.1)$$

Notice that  $\Xi_0^2(X^\bullet, \dots, X^\bullet)(S)$  is the one point space since each Cartesian product  $\prod_{i \in \bar{0}} f^{-1}(i)$  is a point (because  $\bar{0} = \emptyset$ ).

On morphisms, the functor  $\Xi_n^2(X^\bullet, \dots, X^\bullet): \Delta_+ \rightarrow \text{Top}$  is defined in the obvious way. We are now ready to define the operad  $\mathcal{D}_2$ .

**Definition 4.1.** For  $n \geq 0$  the space  $\mathcal{D}_2(n)$  is defined to be

$$\mathcal{D}_2(n) = \text{Tot}(\Xi_n^2(\Delta^\bullet, \dots, \Delta^\bullet)) = \text{Nat}(\Delta^\bullet, \Xi_n^2(\Delta^\bullet, \dots, \Delta^\bullet)).$$

McClure and Smith show [1, Section 9] that the collection  $\{\mathcal{D}_2(n)\}_{n \geq 0}$  is a topological operad. They also show [1, Theorem 9.1 (a)] that this operad is actually weakly equivalent in the category of operads to the operad  $B_2$  of little 2-disks.

**Definition 4.2.** An augmented cosimplicial space  $X^\bullet$  is a  $\Xi^2$ -algebra is endowed with a  $\Xi^2$ -structure if there is a family

$$\{\Theta_n: \Xi_n^2(X^\bullet, \dots, X^\bullet) \rightarrow X^\bullet\}_{n \geq 0}$$

of natural transformations satisfying axioms (a), (b) and (c) of [1, Definition 4.5] with  $\mathcal{F}$  replaced by  $\Xi^2$ .

**Definition 4.3.** An augmented cosimplicial space  $X^\bullet$  is said to be reduced if  $X^0$  is the one point space.

**Proposition 4.4.** [1, Proposition 10.3] A sequence  $\mathcal{O} = \{\mathcal{O}(n)\}_{n \geq 0}$  of topological spaces is endowed with a structure of multiplicative operad if and only if the associated reduced augmented cosimplicial space  $\mathcal{O}^\bullet$  is a  $\Xi^2$ -algebra.

In [1, Theorem 9.1 (b)]  $\mathcal{D}_2$ , McClure and Smith prove that the operad  $\mathcal{D}_2$  acts on  $\text{Tot}\mathcal{O}^\bullet$ , when the reduced augmented cosimplicial space  $\mathcal{O}^\bullet$  is built from a multiplicative operad  $\mathcal{O}$ . We now recall this action. To do that, we will define (for each  $n \geq 0$ ) a map

$$\beta_n: \mathcal{D}_2(n) \times (\text{Tot}\mathcal{O}^\bullet)^n \rightarrow \text{Tot}\mathcal{O}^\bullet.$$

Let  $\alpha = \{\alpha_k\}_{k \geq 0} \in \mathcal{D}_2(n)$ , and let  $(a_1^\bullet, \dots, a_n^\bullet) \in (\text{Tot}\mathcal{O}^\bullet)^n$ . We want to define  $\beta_n(\alpha, (a_1^\bullet, \dots, a_n^\bullet)) \in \text{Tot}\mathcal{O}^\bullet$ . Form the diagram

$$\Delta^\bullet \xrightarrow{\alpha} \Xi_n^2(\Delta^\bullet, \dots, \Delta^\bullet) \xrightarrow{\prod_{i=1}^n a_i^\bullet} \Xi_n^2(\mathcal{O}^\bullet, \dots, \mathcal{O}^\bullet) \xrightarrow{\Theta_n} \mathcal{O}^\bullet$$

in which

- the natural transformation  $\prod_{i=1}^n a_i^\bullet$  is induced by  $a_1^\bullet, \dots, a_n^\bullet$ , and

-  $\Theta_n$  is furnished by the  $\Xi^2$ -structure on  $\mathcal{O}^\bullet$  (we have such a structure by Proposition 4.4).

The natural transformation  $\beta_n(\alpha, (a_1^\bullet, \dots, a_n^\bullet)) : \Delta^\bullet \longrightarrow \mathcal{O}^\bullet$  is then defined to be the composite

$$\beta_n(\alpha, (a_1^\bullet, \dots, a_n^\bullet)) = \Theta_n \circ \prod_{i=1}^n a_i^\bullet \circ \alpha.$$

One can interpret a  $\Xi^2$ -structure in another way. McClure and Smith [1] show that the following definition is equivalent to Definition 4.2.

**Definition 4.5.** *An augmented cosimplicial space  $X^\bullet$  is equipped with a  $\Xi^2$ -structure if for each map  $f : T \longrightarrow \bar{n}$  (here  $T$  is a totally ordered set and  $n \geq 0$ ) with complexity  $\leq 2$ , there exists a map*

$$\langle f \rangle : X^{f^{-1}(1)} \times \dots \times X^{f^{-1}(n)} \longrightarrow X^T$$

*such that the collection of maps  $\{\langle f \rangle\}$  is consistent (see [1, Definition 9.4]), commutative (see [1, Definition 9.5]), associative (see [1, Definition 9.6]) and unital (see [1, Definition 9.7]).*

We are going to interpret (in the new language of the  $\Xi^2$ -structure on  $\mathcal{O}^\bullet$ ) the action of  $\mathcal{D}_2$  on  $\text{Tot}\mathcal{O}^\bullet$ . We need this interpretation because it will be used in the proof of Theorem 5.1 below. Let  $\alpha, a_1^\bullet, \dots, a_n^\bullet$  as before. We want to construct a natural transformation

$$\{\beta_n(\alpha, (a_1^\bullet, \dots, a_n^\bullet))_k : \Delta^k \longrightarrow \mathcal{O}^k\}_{k \geq 0}.$$

So let  $[k]$  be an object of the category  $\Delta$ , and let  $t \in \Delta^k$ . By Definition 4.1,  $\alpha_k$  is a map from  $\Delta^k$  to  $\Xi_n^2(\Delta^\bullet, \dots, \Delta^\bullet)([k])$ . Recalling that this latter space is the colimit of certain  $A_{n[k]}$ -diagram (see (4.1) above), there exists an object

$$[k] \xleftarrow{h} T \xrightarrow{f} \bar{n}$$

in the category  $A_{n[k]}$  such that  $\alpha_k(t)$  is the equivalence class of some  $\tilde{\alpha}_k(t) \in \Delta^{f^{-1}(1)} \times \dots \times \Delta^{f^{-1}(n)}$ . That is,

$$\alpha_k(t) = [\tilde{\alpha}_k(t)].$$

Define  $\beta_n(\alpha, (a_1^\bullet, \dots, a_n^\bullet))_k(t)$  to be the image of  $\tilde{\alpha}_k(t)$  under the composite

$$\begin{array}{ccc} \Delta^{f^{-1}(1)} \times \dots \times \Delta^{f^{-1}(n)} & \xrightarrow{(a_1^{f^{-1}(1)}, \dots, a_n^{f^{-1}(n)})} & \mathcal{O}^{f^{-1}(1)} \times \dots \times \mathcal{O}^{f^{-1}(n)} \xrightarrow{\langle f \rangle} \mathcal{O}^T \\ & \searrow & \downarrow h_* \\ & & \mathcal{O}^k, \end{array}$$

where  $h_*$  is the map induced by  $h$ , and  $\langle f \rangle$  is given by the  $\Xi^2$ -structure of  $\mathcal{O}^\bullet$ , which is itself induced by the multiplicative structure of the operad  $\mathcal{O}$  (we will recall the construction of  $\langle f \rangle$  [1, Section 10] in the following lines). That is,

$$\beta_n(\alpha, (a_1^\bullet, \dots, a_n^\bullet))_k(t) = h_* \circ \langle f \rangle \circ (a_1^{f^{-1}(1)}, \dots, a_n^{f^{-1}(n)})(\tilde{\alpha}_k(t)). \quad (4.2)$$

It is straightforward to check that the map  $\beta_n(\alpha, (a_1^\bullet, \dots, a_n^\bullet))_k : \Delta^k \longrightarrow \mathcal{O}^k$  is well defined. It is also straightforward to check that the collection of maps  $\{\beta_n\}_{n \geq 0}$  defines an action of  $\mathcal{D}_2$  on  $\text{Tot}\mathcal{O}^\bullet$ .

We now recall the construction of  $\langle f \rangle$ . Let  $\mu \in \mathcal{O}(2)$  as in the proof of Theorem 3.11, and let us denote the operad structure of  $\mathcal{O}$  by

$$\gamma : \mathcal{O}(n) \times \mathcal{O}(i_1) \times \dots \times \mathcal{O}(i_n) \longrightarrow \mathcal{O}(i_1 + \dots + i_n).$$

- If  $f: [r+s+1] \rightarrow \bar{2}$  is defined by the word  $f = \underbrace{1 \cdots 1}_{r+1} \underbrace{2 \cdots 2}_{s+1}$ , then  $\langle f \rangle: \mathcal{O}^r \times \mathcal{O}^s \rightarrow \mathcal{O}^{r+s+1}$  is defined by the formula

$$\langle f \rangle(x, y) = \mu(x, d^0 y). \quad (4.3)$$

- If  $f: [2n + i_1 + \cdots + i_n] \rightarrow \overline{n+1}$  is on the form  $f = \underbrace{1 2 \cdots 2}_{i_1+1} \underbrace{1 3 \cdots 3}_{i_2+1} \cdots 1 \underbrace{n+1 \cdots n+1}_{i_n+1} 1$ , then  $\langle f \rangle: \mathcal{O}^n \times \mathcal{O}^{i_1} \times \cdots \times \mathcal{O}^{i_n} \rightarrow \mathcal{O}^{2n+i_1+\cdots+i_n}$  is defined by the formula

$$\langle f \rangle(x, y_1, \dots, y_n) = \gamma(x, d^0 d^{i_1+1} y_1, \dots, d^0 d^{i_n+1} y_n). \quad (4.4)$$

- For a general  $f: T \rightarrow \bar{n}$  of complexity less than or equal to 2, the map  $\langle f \rangle: \mathcal{O}^{f^{-1}(1)} \times \cdots \times \mathcal{O}^{f^{-1}(1)} \rightarrow \mathcal{O}^T$  is defined (as a "combination" of formulas (4.3) and (4.4)) by induction on  $\|T\| = |T| - 1$ . We refer the reader to [1, Section 10] for that induction.

## 5 Proof of Theorem 1.1

The goal of this section is to prove Theorem 1.1 announced in the introduction.

Recalling the definition of the cacti operad  $MS$  from Section 3, we have the following theorem in which the second part is a more precise formulation of Theorem 1.1.

**Theorem 5.1.** (a) *There exists an isomorphism  $q: MS \xrightarrow{\cong} \mathcal{D}_2$ .*

(b) *Let  $\mathcal{O}^\bullet$  be a cosimplicial space defined by a multiplicative operad  $\mathcal{O}$ . Then, for each  $n \geq 0$ , the square*

$$\begin{array}{ccc} MS(n) \times (\text{Tot } \mathcal{O}^\bullet)^n & \xrightarrow{\theta_n} & \text{Tot } \mathcal{O}^\bullet \\ q_n \times id^n \downarrow & & \downarrow id \\ \mathcal{D}_2(n) \times (\text{Tot } \mathcal{O}^\bullet)^n & \xrightarrow{\beta_n} & \text{Tot } \mathcal{O}^\bullet \end{array}$$

*commutes.*

*Proof. Proof of (a).* This part was proved in [2] by P. Salvatore. We will still recall the explicit construction of  $q: MS \rightarrow \mathcal{D}_2$  since we need it to prove part (b). To see that  $q$  is an isomorphism of operads, we will refer the reader to [2, Proposition 8.2].

For each  $n \geq 0$ , we will construct an isomorphism  $q_n: MS(n) \rightarrow \mathcal{D}_2(n)$  in such a way that the collection  $q = \{q_n\}_{n \geq 0}: MS \rightarrow \mathcal{D}_2$  turns out to be a morphism of operads. So let  $n \geq 0$  be an integer.

If  $n = 0$  then  $q_0: * = MS(0) \rightarrow \mathcal{D}_2(0) = *$  is the unique map from the one point space to itself.

If  $n = 1$  then, by (3.4), we have

$$MS(1) = \{\varphi: S^1 \rightarrow S^1 \text{ such that } \varphi \text{ is weakly monotone and } \varphi(*) = *\}.$$

We also have (by Definition 4.1)

$$\mathcal{D}_2(1) = \text{Nat}(\Delta^\bullet, \Xi_1^2(\Delta^\bullet)) = \text{Nat}(\Delta^\bullet, \Delta^\bullet) = \text{Tot } \Delta^\bullet.$$

Let  $\varphi \in MS(1)$ , and let  $t = (-1 \leq t_1 \leq \cdots \leq t_k \leq 1) \in \Delta^k$ . Define  $q_1(\varphi): \Delta^k \rightarrow \Delta^k$  by

$$q_1(\varphi)(t) = (-1 \leq \varphi(t_1) \leq \cdots \leq \varphi(t_k) \leq 1).$$

Now we assume that  $n \geq 2$ . Let  $(x, \tilde{\varphi}) \in MS(n) = \mathcal{I}_2(n) \times \text{Mon}(I, \partial I)$ . Set  $x = (I_1, \dots, I_n)$ . Our goal is to construct

$$q_n(x, \tilde{\varphi}) \in \mathcal{D}_2(n) = \text{Nat}(\Delta^\bullet, \Xi_n^2(\Delta^\bullet, \dots, \Delta^\bullet)).$$

Let  $[k] \in \Delta$ . We want to build a map

$$G_x: \Delta^k \longrightarrow \Xi_n^2(\Delta^\bullet, \dots, \Delta^\bullet)([k]).$$

So let  $t = (-1 \leq t_1 \leq \dots \leq t_k \leq 1) \in \Delta^k$ . Consider families  $\{J_i\}_{i=0}^m$  and  $\{L_l\}_{l=0}^m$  of closed intervals as defined in the beginning of the proof of Theorem 3.11. Let us take back the diagram (see (3.9))

$$[k] \xleftarrow{h} [m] \xrightarrow{f} \bar{n}.$$

Clearly we have  $\text{cplx}(f) \leq 2$  (this is because  $\text{cplx}(x) \leq 2$  by Definition 3.4), and  $h$  is a morphism in the category  $\Delta_+$ . Hence, the triple  $(h, [m], f)$  is an object in the category  $A_{n[k]}$ . Recalling that (by (4.1) above)

$$\Xi_n^2(\Delta^\bullet, \dots, \Delta^\bullet)([k]) = \text{colim}_{A_{n[k]}} F_{n[k]},$$

we are going now to build an explicit element  $G_x([k])(t)$  of the space

$$F_{n[k]}(h, [m], f) = \prod_{i=1}^n \Delta^{|f^{-1}(i)|-1}.$$

Let  $i \in \bar{n}$ . As in the proof of Theorem 3.11, we construct an element  $y_i \in \Delta^{|f^{-1}(i)|-1}$  (see (3.14) for the definition of  $y_i$ ). We thus have an element

$$y = (y_1, \dots, y_n) \in \prod_{i=1}^n \Delta^{|f^{-1}(i)|-1}.$$

Define now  $G_x([k])(t) \in \text{colim}_{A_{n[k]}} F_{n[k]}$  to be the equivalence class of  $y$ . That is,

$$G_x([k])(t) = [(y_1, \dots, y_n)].$$

It is straightforward to check that the family  $G_x = \{G_x([k])\}_{k \geq 0}$  is a natural transformation. The map  $q_n: MS(n) \longrightarrow \mathcal{D}_2(n)$  is then defined by

$$q_n(x, \tilde{\varphi}) = G_x.$$

It is also straightforward to check that the map  $q = \{q_n\}_{n \geq 0}: MS \longrightarrow \mathcal{D}_2$  respects the operad structure.

**Proof of (b).** The result follows immediately when  $n = 0$ .

Let  $n \geq 1$ ,  $a_1^\bullet, \dots, a_n^\bullet \in \text{Tot} \mathcal{O}^\bullet$  and  $(x, \tilde{f}) \in MS(n)$ . We want to show that

$$\beta_n(G_x, (a_1^\bullet, \dots, a_n^\bullet)) = \theta_n((x, \tilde{\varphi}), (a_1^\bullet, \dots, a_n^\bullet)) \in \text{Tot} \mathcal{O}^\bullet.$$

To do that, we will prove the following equality (for each  $k \geq 0$ )

$$\beta_n(G_x, (a_1^\bullet, \dots, a_n^\bullet))_k = \theta_n((x, \tilde{\varphi}), (a_1^\bullet, \dots, a_n^\bullet))_k: \Delta^k \longrightarrow \mathcal{O}^k.$$

Let  $[k] \in \Delta$ . If  $k = 0$  then the desired equality follows.

Now take  $k \geq 1$ , and let  $t = (t_1, \dots, t_k) \in \Delta^k$ . We have

$$\begin{aligned} \beta_n(G_x, (a_1^\bullet, \dots, a_n^\bullet))_k(t) &= h_* \circ \langle f \rangle \circ (a_1^{f^{-1}(1)}, \dots, a_n^{f^{-1}(n)})(y) \text{ by (4.2)} \\ &= h_* \circ \langle f \rangle (a_1^{f^{-1}(1)}(y_1), \dots, a_n^{f^{-1}(n)}(y_n)). \end{aligned}$$

To end the proof of this part, it suffices to get the following equality

$$h_* \circ \langle f \rangle (a_1^{f^{-1}(1)}(y_1), \dots, a_n^{f^{-1}(n)}(y_n)) = \theta_n((x, \tilde{\varphi}), (a_1^\bullet, \dots, a_n^\bullet))_k(t) = \theta_n(f) \quad (5.1)$$

when

$$f = \underbrace{1 \dots 1}_{r+1} \underbrace{2 \dots 2}_{s+1} \quad \text{and} \quad f = 1 \underbrace{2 \dots 2}_{i_1+1} 1 \underbrace{3 \dots 3}_{i_2+1} 1 \dots 1 \underbrace{n+1 \dots n+1}_{i_n+1} 1.$$

- If  $f = \underbrace{1 \cdots 1}_{r+1} \underbrace{2 \cdots 2}_{s+1}$  then the map  $h$  in the diagram  $[r+s] \xleftarrow{h} [r+s+1] \xrightarrow{f} \overline{2}$  is equal to the codegeneracy map  $s^{r+1}$ . Recalling that  $id \in \mathcal{O}(1)$  is the identity operation, we first have

$$\begin{aligned} \langle f \rangle (a_1^r(y_1), a_2^s(y_2)) &= \mu(a_1^r(y_1), d^0 a_2^s(y_2)) \text{ by (4.3)} \\ &= \mu(a_1^r(y_1), \mu(id, a_2^s(y_2))) \in \mathcal{O}^{r+s+1} \text{ by the definition of the coface map } d^0. \end{aligned}$$

Next, recalling that  $e \in \mathcal{O}(0)$  is the distinguish operation in arity 0, we have

$$\begin{aligned} h_*(\langle f \rangle (a_1^r(y_1), a_2^s(y_2))) &= s^{r+1}(\langle f \rangle (a_1^r(y_1), a_2^s(y_2))) \text{ since } h_* = s^{r+1} \\ &= (\mu(a_1^r(y_1), \mu(id, a_2^s(y_2)))) \circ_{r+1} e \text{ by the definition of } s^{r+1} \\ &= \mu(a_1^r(y_1), a_2^s(y_2)) \\ &= \theta_2(f) \in \mathcal{O}^{r+s} \text{ by (3.15),} \end{aligned}$$

thus giving the equality (5.1).

- Now we assume that  $f$  is on the form  $f = 1 \underbrace{2 \cdots 2}_{i_1+1} 1 \underbrace{3 \cdots 3}_{i_2+1} 1 \cdots 1 \underbrace{n+1 \cdots n+1}_{i_n+1} 1$ . We will prove the equality (5.1) when  $n = 3$  and  $f = 1 \underbrace{2 \cdots 2}_{3+1} 1 \underbrace{3 \cdots 3}_{5+1} 1 \underbrace{4 \cdots 4}_{1+1} 1$  for example (the proof being the same for general  $n$  and  $f$ ).

By the formula (4.4), we have

$$\langle f \rangle (a_1^3(y_1), a_2^3(y_2), a_3^5(y_3), a_4^1(y_4)) = \gamma(a_1^3(y_1), d^0 d^4 a_2^3(y_2), d^0 d^6 a_3^5(y_3), d^0 d^2 a_4^1(y_4)) \in \mathcal{O}^{15}$$

Since  $f$  is on the above form, it follows that the map  $h$  in the diagram  $[9] \xleftarrow{h} [15] \xrightarrow{f} \overline{4}$  is defined by the word  $h = 0012333456788899$ . It is not difficult to see that  $h = s^{15} \circ s^{13} \circ s^{12} \circ s^6 \circ s^5 \circ s^1$ . Therefore

$$\begin{aligned} h_*(\langle f \rangle (a_1^3(y_1), a_2^3(y_2), a_3^5(y_3), a_4^1(y_4))) &= \gamma(a_1^3(y_1), a_2^3(y_2), a_3^5(y_3), a_4^1(y_4)) \\ &= ((a_1^3(y_1) \circ_3 a_4^1(y_4)) \circ_2 a_3^5(y_3)) \circ_1 a_2^3(y_2) \\ &= \theta_4(f) \in \mathcal{O}^9 \text{ by (3.17),} \end{aligned}$$

thus completing the proof. □

## References

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